MATH 303 — Measures and Integration
Lecture Notes, Fall 2024

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Contents

CHAPTER 1

Motivating Problems of Measure Theory

1. The Problem of Measurement

A basic (and very old) problem in mathematics is to compute the size (length, area, volume) of geometric objects. Areas of polygons and circles can be computed by elementary methods. More complicated regions bounded by continuous curves can be attacked with methods from calculus. But what about more general subsets of Euclidean space? Does it always make sense to talk about the (hyper-)volume of a subset of $\mathbb{R}^d$? What properties does volume have, and how do we compute it?

We will consider these general questions as the "problem of measurement" in Euclidean space and discuss some approaches to a solution.

2. Riemann Integration and Jordan Content

A good first attempt at solving the problem of measurement comes from the Riemann theory of integration. The basic strategy is to approximate general regions by finite collections of boxes (sets of the form $B = \prod_{i=1}^{d}[a_i, b_i]$). For such a box B, we declare the volume to be $\mathrm{Vol}(B) = \prod_{i=1}^{d}(b_i - a_i)$ and use this to define the volume of more general regions. We will now make this idea rigorous.

Definition 1.1: Darboux Integration

Let $B = \prod_{i=1}^{d}[a_i, b_i]$ be a box in $\mathbb{R}^d$, and let $f : B \to \mathbb{R}$ be a bounded function.

- A *Darboux partition* of B is a family of finite sequences $(x_{i,j})_{1 \le i \le d, 0 \le j \le n_i}$ such that $a_i = x_{i,0} < x_{i,1} < \cdots < x_{i,n_i} = b_i$ for each $i \in \{1, \ldots, d\}$.

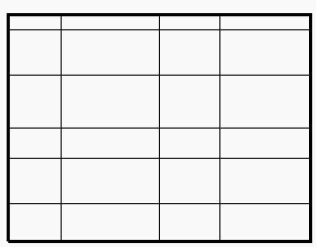

Figure 1.1. A Darboux partition in dimension $d = 2$ with $n_1 = 4$ and $n_2 = 6$.

- Given a Darboux partition $P = (x_{i,j})_{1 \le i \le d, 0 \le j \le n_i}$ of B, the *upper* and *lower Darboux sums of f over B* are given by

$$U_B(f, P) = \sum_{\mathbf{j} \in \prod_{i=1}^{d}\{1,\ldots,n_i\}} \sup_{\boldsymbol{x} \in B_{\mathbf{j}}} f(\boldsymbol{x}) \cdot \mathrm{Vol}(B_{\mathbf{j}})$$

and

$$L_B(f,P) = \sum_{\mathbf{j}\in\prod_{i=1}^d\{1,\ldots,n_i\}} \inf_{\boldsymbol{x}\in B_{\mathbf{j}}} f(\boldsymbol{x})\cdot \mathrm{Vol}(B_{\mathbf{j}}),$$

where $B_{\mathbf{j}}$ is the box $\prod_{i=1}^d [x_{i,j_i-1}, x_{i,j_i}]$, and $\mathrm{Vol}(B_{\mathbf{j}}) = \prod_{i=1}^d (x_{i,j_i} - x_{i,j_i-1})$ is the volume of $B_{\mathbf{j}}$.

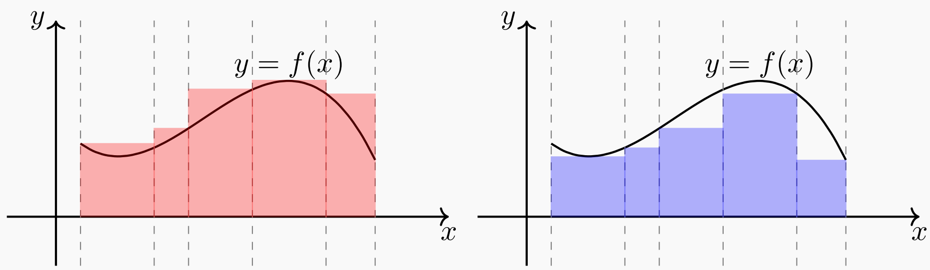

FIGURE 1.2. Upper (red) and lower (blue) Darboux sums of a function f over an interval ($d = 1$).

- The *upper* and *lower Darboux integral of f over B* are

$$U_B(f) = \inf\{U_B(f,P) : P \text{ is a Darboux partition of } B\}$$

and

$$L_B(f) = \sup\{L_B(f,P) : P \text{ is a Darboux partition of } B\}.$$

- The function f is *Darboux integrable over B* if $U_B(f) = L_B(f)$, and their common value is called the *Darboux integral of f over B* and is denoted by $\int_B f(\boldsymbol{x})\, d\boldsymbol{x}$.

PROPOSITION 1.2

A function f is Darboux integrable if and only if it is Riemann integrable. Moreover, the value of the Darboux integral and the Riemann integral (for a Riemann–Darboux integrable function) are the same.

DEFINITION 1.3

A bounded set $E \subseteq \mathbb{R}^d$ is a *Jordan measurable set* if $\mathbb{1}_E$ is Riemann–Darboux integrable over a box containing E. The *Jordan content* of a Jordan measurable set E is the value $J(E) = \int_B \mathbb{1}_E(\boldsymbol{x})\, d\boldsymbol{x}$, where B is any box containing E.

Jordan measurable sets include basic geometric objects such as polyhedra, conic sections, regions bounded by finitely many smooth curves/surfaces, etc.

DEFINITION 1.4

A set $S \subseteq \mathbb{R}^d$ is a *simple set* if it is a finite union of boxes $S = \bigcup_{j=1}^k B_j$.

If the boxes $B_1, \dots, B_k$ are disjoint, then the volume of the simple set $S = \bigcup_{j=1}^k B_j$ is $\mathrm{Vol}(S) = \sum_{j=1}^k \mathrm{Vol}(B_j)$. If some of the boxes intersect, then the volume of $S = \bigcup_{j=1}^k B_j$ can be computed using inclusion-exclusion:

$$\mathrm{Vol}(S) = \sum_{j=1}^{k} \mathrm{Vol}(B_j) - \sum_{1 \le j_1 < j_2 \le k} \mathrm{Vol}(B_{j_1} \cap B_{j_2}) + \sum_{1 \le j_1 < j_2 < j_3 \le k} \mathrm{Vol}(B_{j_1} \cap B_{j_2} \cap B_{j_3}) - \dots$$

This expression is well-defined, since the intersection of two boxes is again a box. A Jordan measurable set is a set that is "well-approximated" by simple sets, as we will make precise now.

Definition 1.5

For a bounded set $E \subseteq \mathbb{R}^d$, define the *inner* and *outer Jordan content* by

$$J_*(E) = \sup \{\mathrm{Vol}(S) : S \subseteq E \text{ is a simple set}\}.$$

and

$$J^*(E) = \inf \{\mathrm{Vol}(S) : S \supseteq E \text{ is a simple set}\}.$$

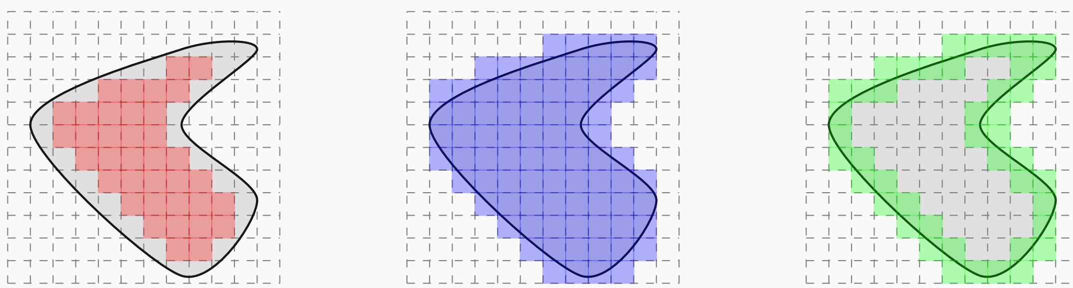

Figure 1.3. Simple sets approximating the inner (red) and outer Jordan content (blue) of a region in dimension $d = 2$. With the red boxes removed from the blue, we get a simple set covering the boundary (in green).

Theorem 1.6

Let $E \subseteq \mathbb{R}^d$ be a bounded set. The following are equivalent:

(i) E is Jordan measurable;
(ii) $J_*(E) = J^*(E)$ (in which case $J(E)$ is equal to this same value);
(iii) $J^*(\partial E) = 0$.

Proof. We will prove the $d = 1$ case. The multidimensional case is similar but more notationally cumbersome, so we omit it to avoid additional technical details that would largely obscure the main ideas.

(i) $\iff$ **(ii)**. To establish this equivalence, it suffices to show

$$U_B(\mathbb{1}_E) = J^*(E) \qquad \text{and} \qquad L_B(\mathbb{1}_E) = J_*(E)$$

for any box (interval) $B \supseteq E$. Let us prove $U_B(\mathbb{1}_E) = J^*(E)$.

Claim 1. $U_B(\mathbb{1}_E) \leq J^*(E)$.

Let $\varepsilon > 0$. Then from the definition of the outer Jordan content, there exists a simple set $S \subseteq \mathbb{R}$ such that $E \subseteq S$ and $\mathrm{Vol}(S) < J^*(E) + \varepsilon$. By assumption, B is an interval containing E, so $S \cap B$ is also a simple set containing E, and $\mathrm{Vol}(S \cap B) \leq \mathrm{Vol}(S) < J^*(E)+\varepsilon$. We may therefore assume without loss of generality that $S \subseteq B$. Write $B = [a, b]$ and $S = [a_1, b_1] \sqcup [a_2, b_2] \sqcup \cdots \sqcup [a_n, b_n]$ with $a \leq a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n \leq b$. We define a Darboux partition[a] P of $[a, b]$ by $P = (x_i)_{i=0}^{2n+1}$ with $x_0 = a$, $x_1 = a_1$, $x_2 = b_1$, $\ldots$, $x_{2n-1} = a_n$, $x_{2n} = b_n$, $x_{2n+1} = b$. Then since $E \subseteq S$, we have

$$\begin{aligned} U_B(\mathbb{1}_E, P) &= \sum_{i=1}^{2n+1} \sup_{x_{i-1} \leq x \leq x_i} \mathbb{1}_E(x) \cdot (x_i - x_{i-1}) \\ &\leq 0 \cdot (a_1 - a) + 1 \cdot (b_1 - a_1) + 0 \cdot (a_2 - b_1) + \cdots + 1 \cdot (b_n - a_n) + 0 \cdot (b - b_n) \\ &= \mathrm{Vol}(S). \end{aligned}$$

Hence, $U_B(\mathbb{1}_E) \leq U_B(\mathbb{1}_E, P) \leq \mathrm{Vol}(S) < J^*(E) + \varepsilon$. This proves the claim.

[a] Strictly speaking, this may fail to be a Darboux partition, since some of the points are allowed to coincide. However, the value we compute for $U_B(\mathbb{1}_E, P)$ will be the correct value for the partition where we remove repetitions of the same point.

Claim 2. $J^*(E) \leq U_B(\mathbb{1}_E)$.

Let $\varepsilon > 0$. Write $B = [a, b]$. Then there exists a Darboux partition $a = x_0 < x_1 < \cdots < x_n = b$ such that $U_B(\mathbb{1}_E, P) < U_B(\mathbb{1}_E) + \varepsilon$. Let $M_i = \sup_{x_{i-1} \leq x \leq x_i} \mathbb{1}_E(x) \in \{0, 1\}$, and note that, by definition, $U_B(\mathbb{1}_E, P) = \sum_{i=1}^n M_i(x_i - x_{i-1})$. Let $I \subseteq \{1, \ldots, n\}$ be the set $I = \{1 \leq i \leq n : M_i = 1\}$, and let $S = \bigcup_{i \in I}[x_{i-1}, x_i]$. Then S is a simple set with length $\mathrm{Vol}(S) = \sum_{i \in I}(x_i - x_{i-1}) = U_B(\mathbb{1}_E, P)$. Moreover, $E \subseteq S$, since S is the union of all intervals that have nonempty intersection with E. Thus, $J^*(E) \leq \mathrm{Vol}(S) = U_B(\mathbb{1}_E, P) < U_B(\mathbb{1}_E) + \varepsilon$.

The identity $L_B(\mathbb{1}_E) = J_*(E)$ is proved similarly.

(ii) $\iff$ **(iii)**. It suffices to prove $J^*(\partial E) = J^*(E) - J_*(E)$. (See Figure 1.3.)

Claim 3. $J^*(\partial E) \leq J^*(E) - J_*(E)$.

Let $\varepsilon > 0$. Let S_1 be a simple set such that $E \subseteq S_1$ and $\mathrm{Vol}(S_1) < J^*(E) + \frac{\varepsilon}{2}$. Since S_1 is closed, we have $\overline{E} \subseteq S_1$. Let S_2 be a simple set with $S_2 \subseteq E$ such that $\mathrm{Vol}(S_2) > J_*(E) - \frac{\varepsilon}{2}$. Note that $\mathrm{int}(S_2) \subseteq \mathrm{int}(E)$. Therefore, $S = S_1 \setminus \mathrm{int}(S_2)$ is a simple set and $\partial E = \overline{E} \setminus \mathrm{int}(E) \subseteq S$, so $J^*(\partial E) \leq \mathrm{Vol}(S) = \mathrm{Vol}(S_2) - \mathrm{Vol}(S_1) < J^*(E) - J_*(E) + \varepsilon$. But ε was arbitrary, so we conclude $J^*(\partial E) \leq J^*(E) - J_*(E)$.

Claim 4. $J^*(E) - J_*(E) \le J^*(\partial E)$.

Let $\varepsilon > 0$, and let $S \supseteq \partial E$ be a simple set with $\mathrm{Vol}(S) < J^*(\partial E) + \frac{\varepsilon}{2}$. Write $S = \bigsqcup_{i=1}^n [a_i, b_i]$ with $a_1 \le b_1 < a_2 \le b_2 < \cdots < a_n \le b_n$. Let $[a,b] \subseteq \mathbb{R}$ such that $E \subseteq [a,b]$ and $a < a_1$ and $b < b_n$. For notational convenience, let $b_0 = a$ and $a_{n+1} = b$. Let $I \subseteq \{0, \ldots, n\}$ be the collection of indices i such that $(b_i, a_{i+1}) \cap E \ne \emptyset$. For each $i \in I$, we claim that $(b_i, a_{i+1}) \subseteq E$. If not, then (b_i, a_{i+1}) contains a boundary point of E, but $\partial E \subseteq S$, so this is a contradiction. Thus, $S' = \bigcup_{i \in I} [b_i, a_{i+1}]$ is a simple set with $\mathrm{int}(S') \subseteq E$. Shrinking slightly each interval in S', we obtain a simple set

$$S'' = \bigcup_{i \in I} \left[b_i + \frac{\varepsilon}{4(n+1)}, a_{i+1} - \frac{\varepsilon}{4(n+1)} \right]$$

such that $S'' \subseteq E$. Moreover, $\mathrm{Vol}(S'') \ge \mathrm{Vol}(S') - \frac{\varepsilon}{2(n+1)}|I| \ge \mathrm{Vol}(S') - \frac{\varepsilon}{2}$. Noting that $S \cup S'$ is a simple set containing E, we arrive at the inequality

$$J^*(E) - J_*(E) \le \mathrm{Vol}(S \cup S') - \mathrm{Vol}(S'') = \mathrm{Vol}(S) + \mathrm{Vol}(S') - \mathrm{Vol}(S'') < J^*(\partial E) + \varepsilon.$$

This completes the proof of Theorem 1.6. □

Example 1.7

The sets $\mathbb{Q} \cap [0,1]$ and $[0,1] \setminus \mathbb{Q}$ are not Jordan measurable (see Exercise 1.1).

In addition to the above example, there are many other "nice" sets that are not Jordan measurable. There are, for instance, bounded open sets in $\mathbb{R}$ that are not Jordan measurable. We will work out one such example in detail.

Example 1.8

The complement U of the fat Cantor set (also known as the Smith–Volterra–Cantor set) $K \subseteq [0,1]$ is Jordan non-measurable. We construct K iteratively, starting from $[0,1]$, by removing intervals of length 4^{-n} at step n. In other words, at step n, we remove an interval of length 4^{-n} around each rational point with denominator 2^n.

Figure 1.4. Iterative construction of the fat Cantor set.

Let

$$U = \bigcup_{n=0}^{\infty} \bigcup_{j=1}^{2^n} \left(\frac{2j+1}{2^{n+1}} - \frac{1}{2 \cdot 4^{n+1}}, \frac{2j+1}{2^{n+1}} + \frac{1}{2 \cdot 4^{n+1}} \right).$$

Then $K = [0,1] \setminus U$. The inner Jordan content of U is

$$J_*(U) = \sum_{n=0}^{\infty} \sum_{j=1}^{2^n} \mathrm{Len} \left(\frac{2j+1}{2^{n+1}} - \frac{1}{2 \cdot 4^{n+1}}, \frac{2j+1}{2^{n+1}} + \frac{1}{2 \cdot 4^{n+1}} \right) = \sum_{n=0}^{\infty} 2^n \cdot \frac{1}{4^{n+1}} = \frac{1}{4} \sum_{n=0}^{\infty} 2^{-n} = \frac{1}{2}.$$

However, $\overline{U} = [0,1]$ (since U contains every rational number whose denominator is a power of 2), so the outer Jordan content of U is $J^*(U) = J^*([0,1]) = 1$.

3. Limits of Integrable Functions

You may recall from the theory of Riemann integration that *uniform* limits of Riemann integrable functions are Riemann integrable, and one may in this case interchange the order of taking limits and computing the integral. More precisely:

Theorem 1.9

Let B be a box in $\mathbb{R}^d$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of Riemann integrable functions on B, and suppose f_n converges uniformly to a function $f : B \to \mathbb{R}$. Then f is Riemann integrable, and

$$\int_B f(\boldsymbol{x})\ d\boldsymbol{x} = \lim_{n\to\infty} \int_B f_n(\boldsymbol{x})\ d\boldsymbol{x}.$$

One of the deficiencies of the Riemann–Darboux–Jordan approach to integration and measurement is that pointwise (non-uniform) limits do not share this property.

Example 1.10

Enumerate the set $\mathbb{Q} \cap [0,1] = \{q_1, q_2, \dots\}$. Let $f_n : [0,1] \to [0,1]$ be the function

$$f_n(x) = \begin{cases} 1, & \text{if } x \in \{q_1, \dots, q_n\} \\ 0, & \text{otherwise.} \end{cases}$$

Then f_n is Riemann integrable and $f_n \to \mathbb{1}_{\mathbb{Q}\cap[0,1]}$ pointwise, but $\mathbb{1}_{\mathbb{Q}\cap[0,1]}$ is not Riemann integrable.

Since analysis so often deals with limits, it is desirable to develop a theory of integration that accommodates pointwise limits. The Lebesgue measure and Lebesgue integral resolve this shortcoming.

4. The Solution of Lebesgue

The Jordan non-measurable set in Example 1.8 appears to have a sensible notion of "length." Indeed, the complement U, being a disjoint union of intervals, could be reasonably assigned as a "length" the sum of the lengths of the (countably many) intervals of which it is made. This produces a value of $\frac{1}{2}$ for the length of U, and so we should take K to also have length $\frac{1}{2}$, since $K \sqcup U = [0,1]$ is an interval of length 1. The feature that U is a disjoint union of intervals turns out to not be any special feature of U at all but instead a general feature of open sets in $\mathbb{R}$.

Proposition 1.11

Let $U \subseteq \mathbb{R}$ be an open set. Then U can be expressed as a countable disjoint union of open intervals.

Proof. Exercise 1.2. □

By Proposition 1.11, it seems reasonable to define the length of an open set $U \subseteq \mathbb{R}$ as follows. Write $U = (a_1, b_1) \sqcup (a_2, b_2) \sqcup \dots$ as a disjoint union of open intervals, and define its length as

$(b_1 - a_1) + (b_2 - a_2) + \ldots$. Then open sets may play the role that simple sets played in the definition of the Jordan content, and this leads to the Lebesgue measure.

REMARK. In higher dimensions, Proposition 1.11 needs to be modified, but one can still reasonably talk about the d-dimensional volume of open sets in $\mathbb{R}^d$. See Exercises 1.3 and 1.6.

DEFINITION 1.12

Let $E \subseteq \mathbb{R}^d$.

- The *outer Lebesgue measure of E* is the quantity

$$\begin{aligned}\lambda^*(E) &= \inf \{\mathrm{Vol}(U) : U \supseteq E \text{ is open}\} \\ &= \inf \left\{ \sum_{j=1}^{\infty} \mathrm{Vol}(B_j) : B_1, B_2, \ldots \text{ are boxes, and } E \subseteq \bigcup_{j=1}^{\infty} B_j \right\}.\end{aligned}$$

- The set E is *Lebesgue measurable* (with *Lebesgue measure* $\lambda(E) = \lambda^*(E)$) if for every $\varepsilon > 0$, there exists an open set $U \subseteq \mathbb{R}^d$ such that $E \subseteq U$ and $\lambda^*(U \setminus E) < \varepsilon$.

PROPOSITION 1.13

If $E \subseteq \mathbb{R}^d$ is Jordan measurable, then E is Lebesgue measurable and $J(E) = \lambda(E)$.

The family of Lebesgue measurable sets is much larger than the family of Jordan measurable sets. Among the several nice properties of the Lebesgue measure (and abstract measures) that we will see later in the course are:

PROPOSITION 1.14

(1) If $(E_n)_{n\in\mathbb{N}}$ are Lebesgue measurable sets, then $\bigcup_{n=1}^{\infty} E_n$ and $\bigcap_{n=1}^{\infty} E_n$ are Lebesgue measurable.
(2) If $(E_n)_{n\in\mathbb{N}}$ are pairwise disjoint and Lebesgue measurable, then $\lambda\left(\bigsqcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \lambda(E_n)$.
(3) If $E_1 \subseteq E_2 \subseteq \cdots \subseteq \mathbb{R}^d$ are Lebesgue measurable sets, then $\lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \lambda(E_n)$.
(4) If $E_1 \supseteq E_2 \supseteq \ldots$ are Lebesgue measurable subsets of $\mathbb{R}^d$ and $\lambda(E_1) < \infty$, then $\lambda\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \lambda(E_n)$.

5. Applications of Abstract Measure Theory

The mathematical language and tools encompassed in measure theory play a foundational role in many other areas of mathematics. A highly abbreviated sampling follows.

PROBABILITY THEORY. Measure theory provides the axiomatic foundations of probability theory, providing rigorous notions of *random variables* and *probabilities* of *events*. Important limit laws (the law of large numbers and central limit theorem, for example) are phrased mathematically using measure-theoretic notions of convergence.

FOURIER ANALYSIS. Periodic (say, continuous or Riemann-integrable) functions on the real line have corresponding Fourier series representations $f(x) \sim \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi i n x}$. The functions $e^{2\pi i n x}$ are orthonormal, and Parseval's identity gives $\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \int_0^1 |f(x)|^2 \, dx$. Given a sequence $(a_n)_{n\in\mathbb{N}}$, one may ask whether $\sum_{n\in\mathbb{Z}} a_n e^{2\pi i n x}$ is the Fourier expansion of some function f, and if so, what properties does f have? Another natural question is whether the series $\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx}$ actually converges to the function f, and if so, in which sense? Both of these questions are properly answered in a measure-theoretic framework. If one is interested in decomposing functions defined on other groups (for instance, on compact abelian groups) into their Fourier series, then one also needs to develop a method of integrating functions on groups in order to compute Fourier coefficients and make sense of Parseval's identity.

Functional Analysis and Operator Theory. When one studies familiar concepts from linear algebra in infinite-dimensional spaces, measures become unavoidable for many tasks. For example, versions of the spectral theorem (generalizing the representation of suitable matrices in terms of their eigenvalues and eigenvectors) for operators on infinite-dimensional spaces require the abstract notion of a measure.

Ergodic Theory. Ergodic theory was developed to study the long-term statistical behavior of dynamical (time-dependent) systems, providing a framework to resolve important problems in physics related to the "ergodic hypothesis" in thermodynamics and the "stability" of the solar system. It turns out that the appropriate mathematical formalism for understanding these problems comes from abstract measure theory.

Fractal Geometry. Self-similar geometric objects such as the Koch snowflake, Sierpiński carpet, and the middle-thirds Cantor set (see Figure 1.5) can be meaningfully assigned a notion of "dimension" that can take a non-integer value. How does one determine the dimension of a fractal object? There are several different approaches to dimension, but one of the most popular is the *Hausdorff dimension*, which relies on a family of measures that interpolate between the integer-dimensional Lebesgue measures.

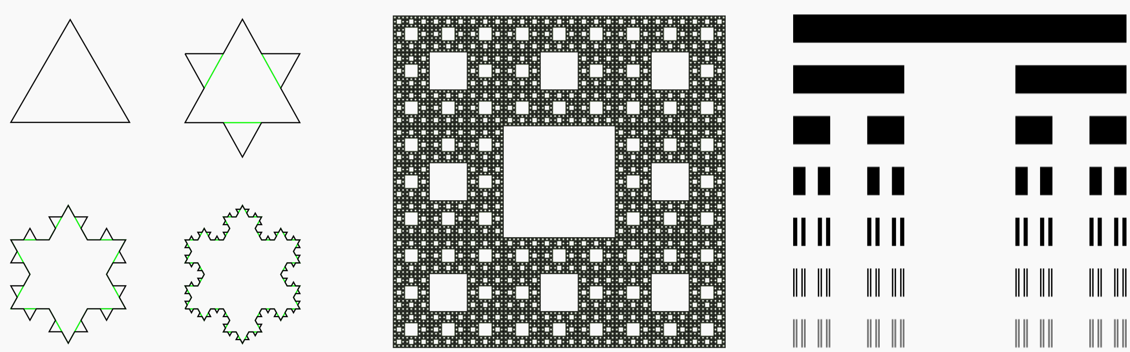

Figure 1.5. Fractal shapes: the Koch snowflake (left) of Hausdorff dimension $\frac{\log 4}{\log 3} \approx 1.26$, Sierpiński carpet (middle) of dimension $\frac{\log 8}{\log 3} \approx 1.89$, and middle-thirds Cantor set (right) of dimension $\frac{\log 2}{\log 3} \approx 0.63$.

Additional Reading

This introductory chapter is heavily influenced by the book of Tao [**6**] on measure theory. Many of the results in this chapter are discussed in greater detail in [**6**, Section 1.1].

Exercises

1.1 Show that $J^*(\mathbb{Q}\cap[0,1]) = J^*([0,1]\setminus\mathbb{Q}) = 1$, and $J_*(\mathbb{Q}\cap[0,1]) = J_*([0,1]\setminus\mathbb{Q}) = 0$.

1.2 Let $U \subseteq \mathbb{R}$ be an open set. Show that U can be written as a disjoint union of countably many open intervals.

1.3 Let $U = \{(x, y) : x^2 + y^2 < 1\} \subseteq \mathbb{R}^2$ be the open unit disk. Show that U cannot be expressed as a disjoint union of countably many open boxes.

1.4 Give an example to show that the statement

$$\lambda^*(E) = \sup_{U \subseteq E, U \text{ open}} \lambda^*(U).$$

is false.

1.5 (Area Interpretation of the Riemann Integral) Let $[a, b]$ be an interval and $f : [a, b] \to [0, \infty)$ a bounded function. Show that f is Riemann integrable if and only if the set

$$E_+ = \{(x, t) : a \leq x \leq b, 0 \leq t \leq f(x)\}$$

is a Jordan measurable set in $\mathbb{R}^2$, in which case

$$\int_a^b f(x)\, dx = J(E_+).$$

1.6 Let $U \subseteq \mathbb{R}^d$ be an open set. Show that U can be written as a disjoint union of countably many half-open boxes (i.e., sets of the form $B = \prod_{i=1}^d [a_i, b_i)$).

CHAPTER 2

Measure Spaces

1. σ-Algebras

Before defining measures, we must determine which subsets of a given set X we would like to be able to measure. The full set X should be measurable, and we should allow ourselves to perform the basic set-theoretic operations (complements, unions, and intersections). Allowing *finite* unions and intersections produces an *algebra* of sets. Algebras are a very useful notion, but (as with the Jordan content discussed in the previous chapter) they are insufficient for appropriately handling limits. We will therefore upgrade from algebras to *σ-algebras*:

Definition 2.1

Let X be a set. A *σ-algebra* on X is a family $\mathcal{B} \subseteq \mathscr{P}(X)$ of subsets of X with the following properties:

- $X \in \mathcal{B}$;
- If $B \in \mathcal{B}$, then $X \setminus B \in \mathcal{B}$;
- If $(B_n)_{n\in\mathbb{N}}$ is a countable family of elements of $\mathcal{B}$, then $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

Remark. In the definition of a σ-algebra, we have made no explicit mention of intersections. However, by De Morgan's laws, we can also generate the countable intersection of sets: $\bigcap_{n\in\mathbb{N}} B_n = X \setminus \left(\bigcup_{n\in\mathbb{N}}(X \setminus B_n)\right)$.

Example 2.2

Some examples of σ-algebras include the following:

- For any set X, the power set $\mathscr{P}(X)$ is a σ-algebra, as is the pair $\{\emptyset, X\}$.
- The family $\mathcal{B} = \{B \subseteq \mathbb{R} : \text{either } B \text{ or } \mathbb{R} \setminus B \text{ is countable}\}$ of countable and co-countable subsets of $\mathbb{R}$ is a σ-algebra.
- Unions of unit-length intervals in $\mathbb{R}$ form a σ-algebra $\mathcal{B} = \left\{\bigcup_{n\in S}[n, n+1) : S \subseteq \mathbb{Z}\right\}$.

Proposition 2.3

Suppose $(\mathcal{B}_i)_{i\in I}$ is a family of σ-algebras on X. Then $\bigcap_{i\in I} \mathcal{B}_i$ is a σ-algebra.

Proof. Let $\mathcal{B} = \bigcap_{i\in I} \mathcal{B}_i$.

For every $i \in I$, we have $X \in \mathcal{B}_i$, so $X \in \mathcal{B}$.

Suppose $B \in \mathcal{B}$. Then $B \in \mathcal{B}_i$ for every $i \in I$, so $X \setminus B \in \mathcal{B}_i$ for every $i \in I$. Hence, $X \setminus B \in \mathcal{B}$.

Let $(B_n)_{n\in\mathbb{N}}$ be a countable family of sets in $\mathcal{B}$. For each $i \in I$, the sets $(B_n)_{n\in\mathbb{N}}$ belong to $\mathcal{B}_i$, so $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}_i$. Therefore, $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$. □

Definition 2.4

The *σ-algebra generated by a family* $\mathcal{S} \subseteq \mathscr{P}(X)$ is the smallest σ-algebra containing $\mathcal{S}$, denoted by $\sigma(\mathcal{S})$.

Remark. Note that $\sigma(\mathcal{S})$ is well-defined by Proposition 2.3:

$$\sigma(\mathcal{S}) = \bigcap \{\mathcal{B} : \mathcal{B} \text{ is a } \sigma\text{-algebra on } X, \mathcal{S} \subseteq \mathcal{B}\}.$$

In topological spaces (such as the real line), we will often consider the σ-algebra generated by the topology.

Definition 2.5

Let (X, τ) be a topological space. The *Borel σ-algebra* is the σ-algebra generated by the open subsets of X, i.e. $\text{Borel}(X) = \sigma(\tau)$.

Borel sets can be placed in a hierarchy in terms of their level of complexity. At the simplest level are the open (G) and closed (F) sets. Next come countable intersections of open sets (G_δ sets) and countable unions of closed sets (F_σ sets) and so on.

$$\begin{array}{ccccccccccccc}
\Sigma^0_1 = G & \xrightarrow{\bigcap} & \Pi^0_2 = G_\delta & \xrightarrow{\bigcup} & \Sigma^0_3 = G_{\delta\sigma} & \xrightarrow{\bigcap} & \Pi^0_4 = G_{\delta\sigma\delta} & \xrightarrow{\bigcup} & \ldots & \xrightarrow{\bigcup} & \Sigma^0_\omega & \xrightarrow{\bigcap} & \ldots \\
\updownarrow c & & \updownarrow c & & \updownarrow c & & \updownarrow c & & & & \updownarrow c & & \\
\Pi^0_1 = F & \xrightarrow{\bigcup} & \Sigma^0_2 = F_\sigma & \xrightarrow{\bigcap} & \Pi^0_3 = F_{\sigma\delta} & \xrightarrow{\bigcup} & \Sigma^0_4 = F_{\sigma\delta\sigma} & \xrightarrow{\bigcap} & \ldots & \xrightarrow{\bigcap} & \Pi^0_\omega & \xrightarrow{\bigcup} & \ldots
\end{array}$$

Figure 2.1. The Borel hierarchy for subsets of a topological space.

The placement of a (Borel) set within the Borel hierarchy is a useful notion of "complexity" for sets. Intuitively speaking, if a set is lower down in the Borel hierarchy, then it is in some sense easier to define than a set higher up the hierarchy. Determining where sets occur in the Borel hierarchy (or if they are Borel at all) is a common theme in an area of mathematical logic known as *descriptive set theory*. We will largely not concern ourselves with such problems in this course, but some suggested additional reading appears at the end of this chapter for those who are interested.

In our development of the abstract theory of measures (where we may not even have a topology to work with), our object of study will be arbitrary sets X equipped with a σ-algebra.

Definition 2.6

A *measurable space* is a pair $(X, \mathcal{B})$, where X is a set and $\mathcal{B}$ is a σ-algebra on X. Elements of the σ-algebra $\mathcal{B}$ are called *measurable sets*.

2. Measurable Functions

Recall that a function $f : X \to Y$ from one topological space to another is continuous if the preimage of every open set in Y is open in X. Measurable functions are defined analogously, but with "open" replaced by "measurable."

Definition 2.7

Let $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ be measurable spaces. A function $f : X \to Y$ is *measurable* if for every $C \in \mathcal{C}$, one has $f^{-1}(C) \in \mathcal{B}$.

Some basic properties of measurable functions that will be used frequently are as follows:

Proposition 2.8

(1) Let $(X, \mathcal{B})$, $(Y, \mathcal{C})$, and $(Z, \mathcal{D})$ be measurable spaces. Let $f : X \to Y$ and $g : Y \to Z$ be measurable functions. Then $g \circ f : X \to Z$ is measurable.
(2) Let $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ be measurable spaces, and let $f : X \to Y$. Suppose $\mathcal{S} \subseteq \mathscr{P}(Y)$ is a family of sets such that $\sigma(\mathcal{S}) = \mathcal{C}$. If $f^{-1}(S) \in \mathcal{B}$ for every $S \in \mathcal{S}$, then f is a measurable function.
(3) Suppose X and Y are topological spaces and $\mathcal{B} = \mathrm{Borel}(X)$ and $\mathcal{C} = \mathrm{Borel}(Y)$ are the Borel σ-algebras on X and Y respectively. Then every continuous function $f : X \to Y$ is measurable.

Proof. (1) Let $D \in \mathcal{D}$. Since g is measurable, we have $C = g^{-1}(D) \in \mathcal{C}$. Then since f is measurable, $B = f^{-1}(C) \in \mathcal{B}$. But $B = f^{-1}(g^{-1}(D)) = (g \circ f)^{-1}(D)$, so $g \circ f$ is measurable.

(2) Let $\mathcal{F} = \{E \subseteq Y : f^{-1}(E) \in \mathcal{B}\}$. We claim that $\mathcal{F}$ is a σ-algebra. Then since $\mathcal{S} \subseteq \mathcal{F}$, we conclude that $\mathcal{C} = \sigma(\mathcal{S}) \subseteq \mathcal{F}$, so f is measurable. Let us now prove the claim:

- $f^{-1}(Y) = X \in \mathcal{B}$, so $Y \in \mathcal{F}$.
- Suppose $E \in \mathcal{F}$. Then $f^{-1}(Y \setminus E) = X \setminus \underbrace{f^{-1}(E)}_{\in \mathcal{B}} \in \mathcal{B}$, so $Y \setminus E \in \mathcal{F}$.
- Suppose $E_1, E_2, \dots \in \mathcal{F}$, and let $E = \bigcup_{n\in\mathbb{N}} E_n$. Then
$$f^{-1}(E) = \bigcup_{n\in\mathbb{N}} \underbrace{f^{-1}(E_n)}_{\in \mathcal{B}} \in \mathcal{B},$$
so $E \in \mathcal{F}$.

This proves that $\mathcal{F}$ is a σ-algebra on Y.

(3) follows from (1) by taking $\mathcal{S}$ to be the collection of open sets in Y. □

3. The Extended Real Numbers and Extended Real-Valued Functions

One obtains an important class of measurable functions when one considers functions defined on a measurable space taking real values. For many applications and in order to account more fully for limits of functions, it is often convenient to work with the slightly more general concept of *extended* real-valued functions.

Definition 2.9

The *extended real numbers* are the set $[-\infty, \infty] = \mathbb{R} \cup \{\infty, -\infty\}$ with the following topological and algebraic properties:

- The topology on $[-\infty, \infty]$ is generated by open intervals (a, b) with $a, b \in \mathbb{R}$ and sets of the form $(a, \infty] = (a, \infty) \cup \{\infty\}$ and $[-\infty, b) = (-\infty, b) \cup \{-\infty\}$ for $a, b \in \mathbb{R}$.

- Addition is extended as a commutative operation with $\infty + x = \infty$ and $-\infty + x = -\infty$ for real numbers $x \in \mathbb{R}$. For addition of two infinite quantities, we define $\infty + \infty = \infty$ and $-\infty + (-\infty) = -\infty$. However, $-\infty + \infty$ is undefined.
- Multiplication is also extended as a commutative operation with the properties

$$x \in (0, \infty) \implies \infty \cdot x = \infty \quad \text{and} \quad -\infty \cdot x = -\infty;$$
$$x \in (-\infty, 0) \implies \infty \cdot x = -\infty \quad \text{and} \quad -\infty \cdot x = \infty.$$

By convention, we define $\infty \cdot 0 = -\infty \cdot 0 = 0$. Multiplication of infinities is defined by $\infty \cdot \infty = (-\infty) \cdot (-\infty) = \infty$, and $-\infty \cdot \infty = -\infty$.

The topology we have defined on $[-\infty, \infty]$ is the *two-point compactification* of $\mathbb{R}$. You will check in the exercises (Exercise 2.1) that $[-\infty, \infty]$ is indeed a compact space (that is homeomorphic to a closed interval, say $[0, 1]$). The algebraic operations on $[-\infty, \infty]$ are all as one would expect, with one exception: $\infty \cdot 0$ is often considered as an "indeterminate form", but here we have given it a definite value of 0. The reason for this convention is the following proposition, which you will also prove in the exercises:

Proposition 2.10

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $[-\infty, \infty]$, and let $c \in \mathbb{R}$. If $(x_n)_{n\in\mathbb{N}}$ converges to an extended real number, then the sequence $(cx_n)_{n\in\mathbb{N}}$ also converges, and

$$\lim_{n\to\infty} (cx_n) = c \cdot \lim_{n\to\infty} x_n. \tag{2.1}$$

Proof. Exercise 2.2. □

In order to have the desirable property (2.1), one has no choice but to define $\infty \cdot 0 = 0$: by taking the sequence $x_n = n$, we have

$$0 \cdot \infty = 0 \cdot \lim_{n\to\infty} n = \lim_{n\to\infty} (0 \cdot n) = 0.$$

Warning: Property (2.1) does not hold for $c \in \{\infty, -\infty\}$, as can be seen by taking a sequence $(x_n)_{n\in\mathbb{N}}$ that converges to 0.

We say that an extended real-valued function $f : X \to [-\infty, \infty]$ defined on a measurable space $(X, \mathcal{B})$ is $\mathcal{B}$*-measurable* (or simply *measurable*) if it is measurable as a function between the measurable spaces $(X, \mathcal{B})$ and $([-\infty, \infty], \text{Borel}([-\infty, \infty]))$. Since we will always take the same σ-algebra on $[-\infty, \infty]$, we omit explicit reference to the Borel σ-algebra when discussing measurable extended real-valued functions.

Proposition 2.11

Let $(X, \mathcal{B})$ be a measurable space.

(1) Let $f : X \to [-\infty, \infty]$. The following are equivalent:
  (a) f is measurable;
  (b) for every $c \in \mathbb{R}$, $f^{-1}((c, \infty]) \in \mathcal{B}$;
  (c) for every $c \in \mathbb{R}$, $f^{-1}([c, \infty]) \in \mathcal{B}$;
  (d) for every $c \in \mathbb{R}$, $f^{-1}([-\infty, c)) \in \mathcal{B}$;
  (e) for every $c \in \mathbb{R}$, $f^{-1}([-\infty, c]) \in \mathcal{B}$.

(2) Suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions from X to $[-\infty,\infty]$. The following functions are also measurable:
   (a) $\sup_{n\in\mathbb{N}} f_n$;
   (b) $\inf_{n\in\mathbb{N}} f_n$;
   (c) $\limsup_{n\to\infty} f_n$;
   (d) $\liminf_{n\to\infty} f_n$.
(3) Suppose $f,g : X \to \mathbb{R}$ are measurable functions. Then $f+g$ and $f\cdot g$ are measurable.

Notation. For convenience, we will often write sets of the form $f^{-1}((c,\infty])$ as $\{f > c\}$ and similarly for $\{f \geq c\}$, $\{f < c\}$, and $\{f \leq c\}$.

Proof of Proposition 2.11. (1) By Proposition 2.8(2), it suffices to check that each of the relevant collections of intervals generates the Borel σ-algebra on $[-\infty,\infty]$. Let us show that the collection of intervals $(c,\infty]$ for $c \in \mathbb{R}$ generates the Borel σ-algebra. All of the other proofs are similar, so we omit them.

Let $\mathcal{S} = \{(c,\infty] : c \in \mathbb{R}\}$. Note that every element of $\mathcal{S}$ is open in $[-\infty,\infty]$, so $\sigma(\mathcal{S}) \subseteq \text{Borel}([-\infty,\infty])$. On the other hand, we can write $(a,b] = (a,\infty] \setminus (b,\infty]$ for $a,b \in \mathbb{R}, a < b$. Every open set in $\mathbb{R}$ is a countable (disjoint) union of such intervals, so every open subset of $\mathbb{R}$ is contained in $\sigma(\mathcal{S})$. We obtain the additional open sets in $[-\infty,\infty]$ from the rays $(c,\infty] \in \mathcal{S}$ and

$$[-\infty, c) = \bigcap_{n\in\mathbb{N}} \left[-\infty, c + \frac{1}{n}\right] = \bigcap_{n\in\mathbb{N}} \left([-\infty,\infty] \setminus \left(c + \frac{1}{n}, \infty\right]\right) \in \sigma(\mathcal{S}).$$

Thus, $\text{Borel}([-\infty,\infty]) \subseteq \sigma(\mathcal{S})$.

(2) We will use (1).

(a) Let $f = \sup_{n\in\mathbb{N}} f_n$. Note that $\{f > c\} = \bigcup_{n\in\mathbb{N}} \{f_n > c\}$. Each of the sets $\{f_n > c\}$ belongs to $\mathcal{B}$, so $\{f > c\} \in \mathcal{B}$.

(b) Similarly to (a), letting $f = \inf_{n\in\mathbb{N}} f_n$, we may express $\{f < c\} = \bigcup_{n\in\mathbb{N}} \underbrace{\{f_n < c\}}_{\in\mathcal{B}} \in \mathcal{B}$.

(c) Recall that $\limsup_{n\to\infty} f_n = \inf_{k\in\mathbb{N}} \sup_{n\geq k} f_n$, so measurability of $\limsup_{n\to\infty} f_n$ follows from (a) and (b).

(d) Similar to (c): $\liminf_{n\to\infty} f_n = \sup_{k\in\mathbb{N}} \inf_{n\geq k} f_n$.

(3) Let $A : \mathbb{R}^2 \to \mathbb{R}$ and $M : \mathbb{R}^2 \to \mathbb{R}$ be the maps $A(x,y) = x+y$ and $M(x,y) = xy$. Both of the maps A and M are continuous and therefore (Borel) measurable. Moreover, $(f+g)(x) = A(f(x),g(x))$ and $(f\cdot g)(x) = M(f(x),g(x))$. Since the composition of measurable maps is measurable (see Proposition 2.8(1)), it suffices to prove $h : x \mapsto (f(x),g(x))$ is a measurable function from X to $\mathbb{R}^2$. By Proposition 2.8(2), we only need to check preimages of sets generating the Borel σ-algebra on $\mathbb{R}^2$. For convenience, we will take the boxes $[a,b)\times[c,d)$ (the first homework problem was to show that every open set in $\mathbb{R}^2$ is a countable (disjoint) union of such boxes, so they generate the Borel σ-algebra). Observe that

$$h^{-1}([a,b)\times[c,d)) = f^{-1}([a,b)) \cap g^{-1}([c,d)) \in \mathcal{B},$$

since f and g are measurable, so h is indeed a measurable function. □

Example 2.12

Let $(X, \mathcal{B})$ be a measurable space and $E \subseteq X$. The function $\mathbb{1}_E$ is measurable if and only if $E \in \mathcal{B}$.

4. Measures

We are now prepared to define measures on abstract measurable spaces.

Definition 2.13

Let $(X, \mathcal{B})$ be a measurable space. A *measure* on $(X, \mathcal{B})$ is a function $\mu : \mathcal{B} \to [0, \infty]$ such that

- $\mu(\emptyset) = 0$;
- COUNTABLE ADDITIVITY: for any sequence $(E_n)_{n\in\mathbb{N}}$ of pairwise disjoint elements of $\mathcal{B}$, one has $\mu\left(\bigsqcup_{n\in\mathbb{N}} E_n\right) = \sum_{n\in\mathbb{N}} \mu(E_n)$.

The triple $(X, \mathcal{B}, \mu)$ is called a *measure space*.

Nontrivial examples of measures take some effort to construct, and we will spend significant portions of the course discussing different methods for constructing interesting measures. However, there are a few immediate examples that do not require complicated constructions.

Example 2.14

Examples of measures include:

- For any set X, the *counting measure* is a measure defined on the σ-algebra $\mathscr{P}(X)$ by $\mu(E) = |E|$ if E is a finite set and $\mu(E) = \infty$ if E is an infinite set.
- Given a point $x \in X$, the *Dirac measure* defined on $\mathscr{P}(X)$ is the measure $\delta_x(E) = 1$ if $x \in E$ and $\delta_x(E) = 0$ if $x \notin E$.

We will use the following basic properties of measures frequently throughout this course:

Proposition 2.15

Let $(X, \mathcal{B}, \mu)$ be a measure space.

(1) MONOTONICITY: For any $A, B \in \mathcal{B}$, if $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

(2) COUNTABLE SUB-ADDITIVITY: For any sequence $(E_n)_{n\in\mathbb{N}}$ in $\mathcal{B}$,

$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) \leq \sum_{n\in\mathbb{N}} \mu(E_n).$$

(3) CONTINUITY FROM BELOW: If $E_1 \subseteq E_2 \subseteq \cdots \in \mathcal{B}$, then

$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

(4) CONTINUITY FROM ABOVE: If $E_1 \supseteq E_2 \supseteq \cdots \in \mathcal{B}$ and $\mu(E_1) < \infty$, then

$$\mu\left(\bigcap_{n\in\mathbb{N}} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

PROOF. (1) Write $B = A \sqcup (B \setminus A)$. Then $\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$, since μ takes nonnegative values.

(2) Define a new sequence of sets E'_n by $E'_1 = E_1$ and $E'_n = E_n \setminus \bigcup_{j=1}^{n-1} E_j$ for $n \geq 2$. Then the sets $(E'_n)_{n\in\mathbb{N}}$ are pairwise disjoint and satisfy $E'_n \subseteq E_n$ and $\bigsqcup_{n\in\mathbb{N}} E'_n = \bigcup_{n\in\mathbb{N}} E_n$. Therefore,

$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \mu\left(\bigsqcup_{n\in\mathbb{N}} E'_n\right) = \sum_{n\in\mathbb{N}} \mu(E'_n) \leq \sum_{n\in\mathbb{N}} \mu(E_n),$$

where in the last step we have applied monotonicity of μ (property (1)).

(3) Let $E'_1 = E_1$ and $E'_n = E_n \setminus E_{n-1}$ for $n \geq 2$. For convenience, we will set $E_0 = \emptyset$ so that we also have $E'_1 = E_1 \setminus E_0$. Then

$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \mu\left(\bigsqcup_{n\in\mathbb{N}} E'_n\right) = \sum_{n\in\mathbb{N}} \mu(E'_n) \overset{(*)}{=} \sum_{n\in\mathbb{N}} (\mu(E_n) - \mu(E_{n-1})) \overset{(**)}{=} \lim_{n\to\infty} \mu(E_n).$$

The step $(*)$ uses additivity of μ, and $(**)$ comes from the telescoping of the sum.

(4) Define a new sequence $A_n = E_1 \setminus E_n$. Then $\emptyset = A_1 \subseteq A_2 \subseteq \dots$, so

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

by (3). But $\bigcup_{n\in\mathbb{N}} A_n = E_1 \setminus \bigcap_{n\in\mathbb{N}} E_n$, so

$$\mu(E_1) - \mu\left(\bigcap_{n\in\mathbb{N}} E_n\right) = \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n) = \mu(E_1) - \lim_{n\to\infty} \mu(E_n),$$

whence we deduce that (4) holds, since $\mu(E_1) < \infty$. □

EXAMPLE 2.16

Property (4) may fail if $\mu(E_1) = \infty$. Let $X = \mathbb{N}$, $\mathcal{B} = \mathscr{P}(\mathbb{N})$, and let μ be the counting measure. Let $E_n = \{m \in \mathbb{N} : m \geq n\}$. Then $\mu(E_n) = \infty$ for every $n \in \mathbb{N}$, but $\bigcap_{n\in\mathbb{N}} E_n = \emptyset$, so

$$\mu\left(\bigcap_{n\in\mathbb{N}} E_n\right) = 0 \neq \infty = \lim_{n\to\infty} \mu(E_n).$$

Additional Reading

The content of this chapter is common to every text on abstract measure theory, though the order of presentation differs. We have elected to follow more or less the order of presentation from Rudin's *Real and Complex Analysis* [4, Chapter 1]. Alternative presentations can be found in [1, Sections 1.2, 1.3, and 2.1], and [6, Section 1.4].

Introductory texts on measure theory tend not to give much treatment to the Borel hierarchy or other topics in descriptive set theory (and we will also not expand on such topics within these lecture notes). Those interested in learning more can take a look at the book of Kechris [2] and/or the lecture notes of Tserunyan [7], which draw quite heavily on [2].

Exercises

2.1 Prove that the extended real line $[-\infty, \infty]$ is homeomorphic to the closed unit interval $[0, 1]$.

2.2 Prove Proposition 2.10.

2.3 Let X, Y be sets and $f : X \to Y$ any function.

(a) Prove that if $\mathcal{C} \subseteq \mathscr{P}(Y)$ is a σ-algebra on Y, then $\mathcal{B} = \{f^{-1}(C) : C \in \mathcal{C}\}$ is a σ-algebra on X.
(b) Prove that for any family of sets $\mathcal{S} \subseteq \mathscr{P}(Y)$, we have $\sigma(f^{-1}(\mathcal{S})) = f^{-1}(\sigma(\mathcal{S}))$.

2.4 Let $(X, \mathcal{B}, \mu)$ be a finite measure space, and let $\mathcal{A} \subseteq \mathscr{P}(X)$ be an algebra on X such that $\sigma(\mathcal{A}) = \mathcal{B}$. Show that for every $B \in \mathcal{B}$ and every $\varepsilon > 0$, there exists $A \in \mathcal{A}$ such that $\mu(A \triangle B) < \varepsilon$.

2.5 Let X be a set. A family of subsets $\mathcal{S} \subseteq \mathscr{P}(X)$ is a *semi-algebra* if

- $\emptyset, X \in \mathcal{S}$;
- if $A, B \in \mathcal{S}$, then $A \cap B \in \mathcal{S}$;
- if $A, B \in \mathcal{S}$, then $A \setminus B = \bigsqcup_{i=1}^{n} C_i$ for some $C_1, \ldots, C_n \in \mathcal{S}$.

Show that if $\mathcal{S}$ is a semi-algebra, then the algebra generated by $\mathcal{S}$ is

$$\mathcal{A}(\mathcal{S}) = \left\{ \bigcup_{i=1}^{n} A_i : n \in \mathbb{N}, A_i \in \mathcal{S} \right\}.$$

Can $\bigcup$ be replaced by $\bigsqcup$?

2.6 Suppose $\mathcal{B}$ is an infinite σ-algebra (on an infinite set X).

(a) Show that $\mathcal{B}$ contains an infinite sequence $(E_n)_{n \in \mathbb{N}}$ of pairwise disjoint sets.
(b) Deduce that $\mathcal{B}$ has at least the cardinality of the continuum.

2.7 Prove that the following sets are Borel sets in $\mathbb{R}$:

(a) The set of points of continuity

$$C_f = \{x \in \mathbb{R} : f \text{ is continuous at } x\}$$

for an arbitrary function $f : \mathbb{R} \to \mathbb{R}$.

(b) The set of points of convergence

$$\mathrm{Conv} = \{x \in \mathbb{R} : \lim_{n \to \infty} f_n(x) \text{ exists}\}$$

for an arbitrary sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$.

2.8 Let $(X, \mathcal{B})$ be a measurable space, and let $\mu : \mathcal{B} \to [0, \infty]$. Prove that μ is a measure if and only if it satisfies the following three properties:

- $\mu(\emptyset) = 0$;
- FINITE ADDITIVITY: for any disjoint sets $A, B \in \mathcal{B}$,
$$\mu(A \sqcup B) = \mu(A) + \mu(B);$$
- CONTINUITY FROM BELOW: if $E_1 \subseteq E_2 \subseteq \cdots \in \mathcal{B}$, then
$$\mu\left(\bigcup_{n \in \mathbb{N}} E_n\right) = \lim_{n \to \infty} \mu(E_n).$$

CHAPTER 3

Integration Against a Measure

Our next task is to develop an integration theory for integrating measurable functions on abstract measure spaces. In the Riemann–Darboux approach to integration, we approximate a function $f : [a, b] \to [0, \infty)$ by step functions, for which we can easily define the integral. For the Lebesgue theory of integration, we will use a similar idea but with a more general class of functions: so-called simple functions.

1. Integration of Simple Functions

Definition 3.1

Let $(X, \mathcal{B})$ be a measurable space. A *simple function* is a measurable function $s : X \to \mathbb{C}$ taking only finitely many values.

Partitioning X into finitely many pieces corresponding to the values of a simple function s, we may write simple functions as linear combinations of indicator functions of measurable sets. That is, $s = \sum_{j=1}^{n} c_j \mathbb{1}_{E_j}$ for some numbers $c_j \in \mathbb{C}$ and measurable sets $E_j \in \mathcal{B}$. Given a measure μ on $(X, \mathcal{B})$, we define the integral of a simple function in the obvious way. To avoid issues with adding and subtracting infinities, we will deal for now only with nonnegative functions.

Definition 3.2

Let $(X, \mathcal{B}, \mu)$ be a measure space and $s : X \to [0, \infty)$ a simple function. Write $s = \sum_{j=1}^{n} c_j \mathbb{1}_{E_j}$ with $c_j \geq 0$ and $E_j \in \mathcal{B}$. The *integral of s with respect to μ* is given by

$$\int_X s \, d\mu = \sum_{j=1}^{n} c_j \mu(E_j).$$

Proposition 3.3

The integral of a nonnegative simple function is well-defined. That is, the value of the integral of a simple function s does not depend on the representation of s as a linear combination of indicator functions of measurable sets.

Proof. Suppose $s = \sum_{j=1}^{n} c_j \mathbb{1}_{E_j}$. Let $a_1, \dots, a_m$ be the finite collection of values taken by s, and let $A_k = \{s = a_k\}$ for $k = 1, \dots, m$. Then the sets $A_1, \dots, A_k$ partition X, and $s = \sum_{k=1}^{m} a_k \mathbb{1}_{A_k}$. We will show $\sum_{j=1}^{n} c_j \mu(E_j) = \sum_{k=1}^{m} a_k \mu(A_k)$.

Define a new collection of sets $E'_J = \bigcap_{j \in J} E_j \setminus \bigcup_{i \notin J} E_j$ for $J \subseteq \{1, \dots, n\}$. In other words, $x \in E'_J$ means that $x \in E_j$ if and only if $j \in J$. This defines a partition of X. Note that the value of s on the set E'_J is $c'_J = \sum_{j \in J} c_j$. We can therefore relate the sets E'_J to the sets A_k by

$$A_k = \bigsqcup_{J \subseteq \{1,\dots,n\}, c'_J = a_k} E'_J.$$

Then on the one hand,

$$\sum_{k=1}^m a_k \mu(A_k) = \sum_{k=1}^m a_k \sum_{J \subseteq \{1,\dots,n\}, c'_J = a_k} \mu(E'_J) = \sum_{J \subseteq \{1,\dots,n\}} c'_J \mu(E'_J).$$

On the other hand,

$$\sum_{j=1}^n c_j \mu(E_j) = \sum_{j=1}^n c_j \sum_{\{j\} \subseteq J \subseteq \{1,\dots,n\}} \mu(E'_J) = \sum_{J \subseteq \{1,\dots,n\}} \sum_{j \in J} c_j \mu(E'_J) = \sum_{J \subseteq \{1,\dots,n\}} c'_J \mu(E'_J).$$

This completes the proof. □

We used a particular representation of a simple function in the previous proof that will continue to be convenient to work with. Say that $\sum_{j=1}^n c_j \mathbb{1}_{E_j}$ is the *standard representation* of a simple function s if $s = \sum_{j=1}^n c_j \mathbb{1}_{E_j}$, and the sets $E_1, \dots, E_n$ partition X (that is, they are pairwise disjoint and their union is X).

Proposition 3.4

Let $(X, \mathcal{B}, \mu)$ be a measure space, let $s, t : X \to [0, \infty)$ be simple functions, and let $c \in \mathbb{R}$, $c \geq 0$. Then

(1) $\int_X cs \; d\mu = c \cdot \int_X s \; d\mu$;
(2) $\int_X (s + t) \; d\mu = \int_X s \; d\mu + \int_X t \; d\mu$;
(3) if $s \leq t$, then $\int_X s \; d\mu \leq \int_X t \; d\mu$.

Proof. (1) Let $s = \sum_{j=1}^n c_j \mathbb{1}_{E_j}$. Then $cs = \sum_{j=1}^n (cc_j) \mathbb{1}_{E_j}$, so

$$\int_X cs \; d\mu = \sum_{j=1}^n (cc_j) \mu(E_j) = c \cdot \sum_{j=1}^n c_j \mu(E_j) = c \cdot \int_X s \; d\mu.$$

For (2) and (3), it will be helpful to work with the standard representation, so let $s = \sum_{j=1}^n c_j \mathbb{1}_{E_j}$ and $t = \sum_{k=1}^m d_k \mathbb{1}_{F_k}$ be the standard representations. Define sets $A_{j,k} = E_j \cap F_k$ for $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$. Then $E_j = \bigsqcup_{k=1}^m A_{j,k}$ and $F_k = \bigsqcup_{j=1}^n A_{j,k}$.

(2) The function $s + t$ takes the value $c_j + d_k$ on $A_{j,k}$, so

$$\int_X (s+t) \; d\mu = \sum_{j,k} (c_j + d_k) \mu(A_{j,k}) = \sum_{j=1}^n c_j \underbrace{\sum_{k=1}^m \mu(A_{j,k})}_{\mu(E_j)} + \sum_{k=1}^m d_k \underbrace{\sum_{j=1}^n \mu(A_{j,k})}_{\mu(F_k)} = \int_X s \; d\mu + \int_X t \; d\mu.$$

(3) By assumption, if $A_{j,k} \neq \emptyset$, then $c_j \leq d_k$. Thus,

$$\int_X s \; d\mu = \sum_{j=1}^n c_j \mu(E_j) = \sum_{j,k} c_j \mu(A_{j,k}) \leq \sum_{j,k} d_k \mu(A_{j,k}) = \sum_{k=1}^m d_k \mu(F_k) = \int_X t \; d\mu.$$

□

Definition 3.5

Let $(X, \mathcal{B}, \mu)$ be a measure space, $s : X \to [0, \infty)$ a simple function, and $E \in \mathcal{B}$ a measurable set. The *integral of s with respect to μ over E* is given by

$$\int_E s \, d\mu = \int_X s \cdot \mathbb{1}_E \, d\mu.$$

Note that if s is simple, then $s \cdot \mathbb{1}_E$ is also simple, so the above definition makes sense.

Proposition 3.6

Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $s : X \to [0, \infty)$ be a simple function. Then

$$\nu(E) = \int_E s \, d\mu$$

defines a measure on $(X, \mathcal{B})$.

Proof. Note that $s \cdot \mathbb{1}_\emptyset = 0$, so $\nu(\emptyset) = 0$. Suppose $(E_n)_{n\in\mathbb{N}}$ is a pairwise disjoint family of measurable sets, and let $E = \bigsqcup_{n\in\mathbb{N}} E_n$. Write $s = \sum_{j=1}^m a_j \mathbb{1}_{A_j}$. Then $s \cdot \mathbb{1}_E = \sum_{j=1}^m a_j \mathbb{1}_{A_j \cap E}$, so

$$\nu(E) = \sum_{j=1}^m a_j \mu(A_j \cap E) = \sum_{j,n} a_j \mu(A_j \cap E_n) = \sum_{n\in\mathbb{N}} \int_X s \cdot \mathbb{1}_{E_n} \, d\mu = \sum_{n\in\mathbb{N}} \nu(E_n).$$

Note that the sum over n is an infinite sum so reordering requires some justification. Fortunately, all of the values $a_j \mu(A_j \cap E_n)$ are nonnegative, so the sum can be computed in any order without changing the value. □

2. Integration of Nonnegative Measurable Functions

We now want to extend the definition of the integral against a measure to all nonnegative measurable functions. The next proposition shows that simple functions are a sufficiently general class to approximate arbitrary measurable functions.

Proposition 3.7

Let $(X, \mathcal{B})$ be a measurable space, and let $f : X \to [0, \infty]$ be measurable. Then there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of simple functions such that $0 \le s_1 \le s_2 \le \cdots \le f$, and $s_n \to f$ pointwise.

Proof. For $n \in \mathbb{N}$, define

$$s_n(x) = \begin{cases} \frac{a}{2^n}, & \text{if } \frac{a}{2^n} \le f(x) < \frac{a+1}{2^n} \text{ and } a < n \cdot 2^n. \\ n, & \text{if } f(x) \ge n. \end{cases}$$

□

It is therefore reasonable to define the integral of an arbitrary nonnegative measurable function as follows.

Definition 3.8

Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $f : X \to [0, \infty]$ be measurable. We define the *integral of f with respect to μ* as

$$\int_X f \, d\mu = \sup \left\{ \int_X s \, d\mu : s \text{ simple and } 0 \leq s \leq f \right\}.$$

Given a measurable set $E \in \mathcal{B}$, the *integral of f with respect to μ over E* is defined by

$$\int_E f \, d\mu = \int_X f \cdot \mathbb{1}_E \, d\mu.$$

One may object at this point and suggest an alternative definition. Since $f : X \to [0, \infty]$ can be obtained as $f = \lim_{n\to\infty} s_n$ for an increasing sequence of simple functions $0 \leq s_1 \leq s_2 \leq \dots$, why not define $\int_X f \, d\mu = \lim_{n\to\infty} \int_X s_n \, d\mu$? As we will see shortly, this is in fact an equivalent definition that is extremely useful for many applications. However, *as a definition*, it has two serious defects: why should the limit exist? and why should the value be the same for all possible approximations by simple functions? This is why we prefer Definition 3.8 above (and why this is the standard definition across measure theory textbooks).

Proposition 3.9

Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $f, g : X \to [0, \infty]$ be measurable. If $f \leq g$, then

$$\int_X f \, d\mu \leq \int_X g \, d\mu.$$

Proof. It suffices to observe $\{s \text{ simple function} : 0 \leq s \leq f\} \subseteq \{s \text{ simple function} : 0 \leq s \leq g\}$. □

Theorem 3.10: Monotone Convergence Theorem

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions $0 \leq f_1 \leq f_2 \leq \dots$, and let $f = \lim_{n\to\infty} f_n$. Then

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$$

Remark. Note that a consequence of the monotone convergence theorem is that $\int_X f \, d\mu$ can be computed by taking a sequence of simple functions $0 \leq s_1 \leq s_2 \leq \cdots \to f$ and computing $\lim_{n\to\infty} \int_X s_n \, d\mu$.

Proof of Monotone Convergence Theorem. First, f is a measurable function by Proposition 2.11. By monotonicity of the integral (Proposition 3.9), the sequence $\int_X f_n \, d\mu$ is increasing, so $\lim_{n\to\infty} \int_X f_n \, d\mu = \sup_{n\in\mathbb{N}} \int_X f_n \, d\mu \in [0, \infty]$ exists as an extended real number. Moreover,

$$\int_X f \, d\mu \geq \lim_{n\to\infty} \int_X f_n \, d\mu,$$

since the inequality holds for each $n \in \mathbb{N}$. Therefore, it suffices to show

$$\int_X f \, d\mu \leq \lim_{n\to\infty} \int_X f_n \, d\mu.$$

If $\lim_{n\to\infty}\int_X f_n\, d\mu = \infty$, there is nothing to prove, so assume $\lim_{n\to\infty}\int_X f_n\, d\mu < \infty$.

Let $c < 1$. Let $s : X \to [0, \infty)$ be a simple function, $0 \le s \le f$. For $n \in \mathbb{N}$, let $E_n = \{f_n \ge cs\}$. Then $E_1 \subseteq E_2 \subseteq \dots$ and $X = \bigcup_{n\in\mathbb{N}} E_n$. By Proposition 3.6, let $\nu : \mathcal{B} \to [0, \infty]$ be the measure $\nu(E) = \int_E s\, d\mu$. We have

$$\begin{aligned} c \cdot \int_X s\, d\mu &= c \cdot \nu(X) \\ &= c \cdot \lim_{n\to\infty} \nu(E_n) && \text{(continuity from below)} \\ &= \lim_{n\to\infty} c \cdot \nu(E_n) && \text{(Proposition 2.10)} \\ &= \lim_{n\to\infty} \int_{E_n} cs\, d\mu && \text{(Proposition 3.4)} \\ &\le \lim_{n\to\infty} \int_X f_n\, d\mu && \text{(monotonicity).} \end{aligned}$$

Taking a supremum over all such simple functions, we conclude

$$c \cdot \int_X f\, d\mu \le \lim_{n\to\infty} \int_X f_n\, d\mu.$$

Letting $c \to 1$ yields the desired result. □

Proposition 3.11

Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $f, g : X \to [0, \infty]$ be measurable functions. Let $c \in [0, \infty)$.

(1) $\int_X cf\, d\mu = c \cdot \int_X f\, d\mu$.

(2) $\int_X (f + g)\, d\mu = \int_X f\, d\mu + \int_X g\, d\mu$.

Proof. (1) follows quickly from the definition of the integral and Proposition 3.4.

For (2), we will use the monotone convergence theorem. Let $0 \le s_1 \le s_n \le \cdots \le f$ with $s_n \to f$ and $0 \le t_1 \le t_2 \le \cdots \le g$ with $t_n \to g$. Then $0 \le s_1 + t_1 \le s_2 + t_2 \le \cdots \le f + g$ and $s_n + t_n \to f + g$. Thus,

$$\begin{aligned} \int_X (f+g)\, d\mu &= \lim_{n\to\infty} \int_X (s_n + t_n)\, d\mu && \text{(MCT)} \\ &= \lim_{n\to\infty} \int_X s_n\, d\mu + \lim_{n\to\infty} \int_X t_n\, d\mu && \text{(Proposition 3.4)} \\ &= \int_X f\, d\mu + \int_X g\, d\mu && \text{(MCT).} \end{aligned}$$

□

Theorem 3.12

Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative measurable functions, $f_n : X \to [0, \infty]$. Then

$$\int_X \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \sum_{n=1}^{\infty} \int_X f_n\, d\mu.$$

PROOF. We have

$$\begin{aligned}
\int_X \left(\sum_{n=1}^{\infty} f_n\right) \, d\mu &= \int_X \lim_{N\to\infty} \left(\sum_{n=1}^{N} f_n\right) \, d\mu \\
&= \lim_{N\to\infty} \int_X \left(\sum_{n=1}^{N} f_n\right) \, d\mu && \text{(MCT)} \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \int_X f_n \, d\mu && \text{(additivity of the integral)} \\
&\sum_{n=1}^{\infty} \int_X f_n \, d\mu.
\end{aligned}$$

□

THEOREM 3.13: FATOU'S LEMMA

Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions, $f_n : X \to [0, \infty]$. Then

$$\int_X \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

PROOF. Let $f = \liminf_{n\to\infty} f_n$. Define $F_N = \inf_{n\geq N} f_n$. Then $0 \leq F_1 \leq F_2 \leq \dots$ and $F_N \to f$. Therefore,

$$\begin{aligned}
\int_X f \, d\mu &= \lim_{N\to\infty} \int_X F_N \, d\mu && \text{(MCT)} \\
&\leq \lim_{N\to\infty} \inf_{n\geq N} \int_X f_n \, d\mu && \text{(monotonicity of the integral)} \\
&= \liminf_{N\to\infty} \int_X f_n \, d\mu.
\end{aligned}$$

□

3. Integration of Real and Complex-Valued Functions

The method for integrating real and complex-valued functions involves decomposing these functions as linear combinations of nonnegative functions. An important observation is that such a decomposition can be done in a measurable way.

DEFINITION 3.14

Let X be a set and $f : X \to [-\infty, \infty]$. The *positive part* f^+ and *negative part* f^- of f are defined by

$$f^+ = \max\{f, 0\} \qquad \text{and} \qquad f^- = \max\{-f, 0\}.$$

Note that $f = f^+ - f^-$ and $|f| = f^+ + f^-$. Moreover, if $(X, \mathcal{B})$ is a measurable space and $f : X \to [-\infty, \infty]$ is measurable, then f^+ and f^- are measurable by Proposition 2.11.

Definition 3.15

Let $(X, \mathcal{B}, \mu)$ be a measure space.

- An extended real-valued measurable function $f : X \to [-\infty, \infty]$ is *integrable* if
$$\int_X |f| \, d\mu < \infty.$$
In this case, the *integral of f* is defined by
$$\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu.$$
- A complex-valued measurable function $f : X \to \mathbb{C}$ is *integrable* if
$$\int_X |f| \, d\mu < \infty,$$
and the *integral of f* is defined by
$$\int_X f \, d\mu = \int_X \operatorname{Re}(f) \, d\mu + i \int_X \operatorname{Im}(f) \, d\mu.$$
- Given a measurable set $E \in \mathcal{B}$, a measurable function f taking extended real or complex values is *integrable over E* if $f \cdot \mathbb{1}_E$ is integrable, and the *integral of f over E* is
$$\int_E f \, d\mu = \int_X f \cdot \mathbb{1}_E \, d\mu.$$

Remark. By monotonicity of the integral (Proposition 3.9), if a function is integrable, then it is also integrable over every measurable subset of X.

4. Integral Identities and Inequalities

Proposition 3.16: Triangle Inequality for the Integral

Suppose $(X, \mathcal{B}, \mu)$ is a measure space and $f : X \to \mathbb{C}$ is an integrable function. Then
$$\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\mu.$$

Proof. First, suppose f is real-valued. Then by the triangle inequality and linearity,
$$\left| \int_X f \, d\mu \right| = \left| \int_X f^+ \, d\mu - \int_X f^- \, d\mu \right| \leq \int_X f^+ \, d\mu + \int_X f^- \, d\mu = \int_X |f| \, d\mu.$$

Now suppose f is complex-valued. Let $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $\left| \int_X f \, d\mu \right| = \lambda \int_X f \, d\mu$. Then
$$\left| \int_X f \, d\mu \right| = \operatorname{Re}\left(\int_X \lambda f \, d\mu \right) = \int_X \operatorname{Re}(\lambda f) \, d\mu \leq \int_X |\operatorname{Re}(\lambda f)| \, d\mu \leq \int_X |f| \, d\mu.$$

□

Proposition 3.17: Linearity of the Integral

Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $f, g : X \to \mathbb{C}$ be integrable functions, and let $c \in \mathbb{C}$. Then

(1) $f + g$ is integrable, and $\int_X (f+g)\, d\mu = \int_X f\, d\mu + \int_X g\, d\mu$.

(2) cf is integrable, and $\int_X cf\, d\mu = c \int_X f\, d\mu$.

Proof. (1) First, by the triangle inequality, we have $|f+g| \leq |f| + |g|$. Therefore,

$$\int_X |f+g|\, d\mu \overset{(*)}{\leq} \int_X (|f| + |g|)\, d\mu \overset{(**)}{=} \int_X |f|\, d\mu + \int_X |g|\, d\mu < \infty.$$

In step $(*)$, we have used monotonicity of the integral (Proposition 3.9), and in $(**)$, we have used additivity (Proposition 3.11).

Decomposing f and g into their real and imaginary parts, it suffices to prove the identity $\int_X (f+g)\, d\mu = \int_X f\, d\mu + \int_X g\, d\mu$ for real-valued functions f and g. Let $h = f + g$. Then $h = h^+ - h^- = f^+ - f^- + g^+ - g^-$. This can be rearranged to the identity $h^+ + f^- + g^- = h^- + f^+ + g^+$. Then using additivity of the integral for nonnegative functions (Proposition 3.11), we have

$$\begin{aligned} \int_X h^+\, d\mu + \int_X f^-\, d\mu + \int_X g^-\, d\mu &= \int_X (h^+ + f^- + g^-)\, d\mu \\ &= \int_X (h^- + f^+ + g^+)\, d\mu \qquad (3.1) \\ &= \int_X h^-\, d\mu + \int_X f^+\, d\mu + \int_X g^+\, d\mu. \end{aligned}$$

Rearranging again,

$$\begin{aligned} \int_X (f+g)\, d\mu &= \int_X h^+\, d\mu - \int_X h^-\, d\mu && \text{(Definition 3.15)} \\ &= \int_X f^+\, d\mu - \int_X f^-\, d\mu + \int_X g^+\, d\mu - \int_X g^-\, d\mu && \text{(by (3.1))} \\ &= \int_X f\, d\mu + \int_X g\, d\mu && \text{(Definition 3.15)} \end{aligned}$$

(2) Note that $|cf| = |c||f|$, so

$$\int_X |cf|\, d\mu = \int_X |c||f|\, d\mu \overset{(*)}{=} |c| \int_X |f|\, d\mu < \infty,$$

where $(*)$ follows from Proposition 3.11. Hence, cf is integrable.

For computing the integral of cf, we consider several different cases.

Case 1. $c \geq 0$

When f is nonnegative, we have

$$\int_X cf\ d\mu = c \int_X f\ d\mu$$

by Proposition 3.11. The identity follows for a general complex-valued function f by decomposing $f = \left(\operatorname{Re}(f)^+ - \operatorname{Re}(f)^-\right) + i\left(\operatorname{Im}(f)^+ - \operatorname{Im}(f)^-\right)$.

Case 2. $c = -1$

For real-valued $f : X \to \mathbb{R}$, we use the identities $(-f)^+ = f^-$ and $(-f)^- = f^+$ to obtain

$$\int_X (-f)\ d\mu = \int_X f^-\ d\mu - \int_X f^+\ d\mu = -\int_X f\ d\mu.$$

Complex-valued functions can be handled by decomposing into real and imaginary parts.

Case 3. $c = i$

Noting that $\operatorname{Re}(if) = -\operatorname{Im}(f)$ and $\operatorname{Im}(if) = \operatorname{Re}(f)$, we have

$$\begin{aligned}
\int_X if\ d\mu &= \int_X (-\operatorname{Im}(f))\ d\mu + i\int_X \operatorname{Re}(f)\ d\mu && \text{(Definition 3.15)}\\
&= -\int_X \operatorname{Im}(f)\ d\mu + i\int_X \operatorname{Re}(f)\ d\mu && \text{(Case 2)}\\
&= i\left(\int_X \operatorname{Re}(f)\ d\mu + i\int_X \operatorname{Im}(f)\ d\mu\right)\\
&= i\int_X f\ d\mu && \text{(Definition 3.15)}
\end{aligned}$$

Case 4. $c \in \mathbb{R}$

Combine Case 1 and Case 2.

Case 5. $c \in \mathbb{C}$

Write $c = a + ib$ with $a, b \in \mathbb{R}$. Then

$$\begin{aligned}
\int_X cf \, d\mu &= \int_X (af + ibf) \, d\mu \\
&= \int_X af \, d\mu + \int_X ibf \, d\mu && \text{(by (1))} \\
&= \int_X af \, d\mu + i \int_X bf \, d\mu && \text{(Case 3)} \\
&= a \int_X f \, d\mu + ib \int_X f \, d\mu && \text{(Case 4)} \\
&= c \int_X f \, d\mu.
\end{aligned}$$

□

Let $(X, \mathcal{B}, \mu)$ be a measure space, and denote by $L^1(\mu)$ the set of integrable functions. Proposition 3.17 shows that $L^1(\mu)$ is a (complex) vector space. Moreover, in the course of the proof, we showed

$$\int_X |cf| \, d\mu = |c| \int_X |f| \, d\mu \qquad \text{and} \qquad \int_X |f + g| \, d\mu \leq \int_X |f| \, d\mu + \int_X |g| \, d\mu.$$

In other words, if we let

$$\|f\|_1 = \int_X |f| \, d\mu,$$

then $\|\cdot\|_1$ defines a *seminorm* on the vector space of integrable functions on $(X, \mathcal{B}, \mu)$.

Definition 3.18

Let V be a real or complex vector space. A function $\|\cdot\| : V \to [0, \infty)$ is a *seminorm* if it satisfies:

- Triangle inequality: $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in V$, and
- Absolute homogeneity: $\|cv\| = |c| \, \|v\|$ for all $v \in V$ and all scalars c.

A seminorm is a *norm* if it satisfies the additional property

- Positive definite: if $v \in V$ and $\|v\| = 0$, then $v = 0$.

The seminorm $\|\cdot\|_1$ on the space of integrable functions may not be a norm in general, but a small modification will turn it into a norm. This will be discussed in greater detail later in the course, in the context of so-called L^p spaces. One of the important ingredients is a deeper understanding of *null sets*, which we will discuss now.

5. Sets of Measure Zero

Definition 3.19

Let $(X, \mathcal{B}, \mu)$ be a measure space.

- A measurable set $N \in \mathcal{B}$ is a *null set* if $\mu(N) = 0$.
- We say that a property holds *almost everywhere* if there exists a null set $N \in \mathcal{B}$ such that the property holds for every point $x \in X \setminus N$.

Remark. An easy consequence of countable additivity and monotonicity of measures is that the family $\mathcal{N}$ of null sets forms a σ-*ideal* of $\mathcal{B}$:

- $\emptyset \in \mathcal{N}$;
- if $A \in \mathcal{N}$ and $B \in \mathcal{B}$ with $B \subseteq A$, then $B \in \mathcal{N}$; and
- if $(N_n)_{n\in\mathbb{N}}$ is a countable family of null sets, then $\bigcup_{n\in\mathbb{N}} N_n \in \mathcal{N}$.

Notation. The phrases "almost everywhere" or "almost every" are often abbreviated by a.e. or μ-a.e. if the measure needs to be specified. In a statement of the form "Property P holds a.e.," we interpret a.e. as "almost everywhere." For a statement of the form "Property P holds for a.e. $x \in X$," we read a.e. as "almost every," and the meaning is the same as in the previous example statement.

Null sets naturally arise and play an important role in integration theory. Some examples are provided by the next three propositions.

Proposition 3.20

Let $(X, \mathcal{B}, \mu)$ be a measure space. Suppose $f : X \to [-\infty, \infty]$ is an integrable function. Then $f(x) \in \mathbb{R}$ for μ-a.e. $x \in X$.

Proof. Let $N = \{x \in \mathbb{R} : |f(x)| = \infty\}$. We want to show that N is a null set. By monotonicity of the integral (Proposition 3.9),

$$\int_X |f| \; d\mu \geq \int_N |f| \; d\mu = \infty \cdot \mu(N).$$

On the other hand, by integrability of f,

$$\int_X |f| \; d\mu < \infty.$$

Thus, $\infty \cdot \mu(N) < \infty$, so $\mu(N) = 0$. □

Proposition 3.21

Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $f, g : X \to \mathbb{C}$ be measurable functions. Suppose $f = g$ a.e. Then f is integrable if and only if g is integrable. Moreover, if f and g are integrable, then

$$\int_X f \; d\mu = \int_X g \; d\mu.$$

Proof. Let $N = \{x \in X : f(x) \neq g(x)\}$. By assumption, N is a null set.

Step 1. Integrability

Suppose f is integrable. Then

$$\begin{aligned}\int_X |g|\ d\mu &= \int_{X\setminus N} |f|\ d\mu + \int_N |g|\ d\mu && \text{(linearity of the integral)}\\ &\le \int_X |f|\ d\mu + \underbrace{\infty \cdot \mu(N)}_{0} && \text{(monotonicity of the integral)}\\ &= \int_X |f|\ d\mu < \infty,\end{aligned}$$

so g is integrable. Reversing the roles of f and g proves the converse.

STEP 2. Integral

Assume f and g are integrable. Then

$$\begin{aligned}\left|\int_X g\ d\mu - \int_X f\ d\mu\right| &= \left|\int_X (g-f)\ d\mu\right| && \text{(linearity of the integral)}\\ &\le \int_X |g-f|\ d\mu && \text{(triangle inequality for the integral)}\\ &= \int_{X\setminus N} 0\ d\mu + \int_N |g-f|\ d\mu && \text{(linearity of the integral)}\\ &\le 0\cdot\mu(X\setminus N) + \infty\cdot\mu(N) = 0.\end{aligned}$$

□

PROPOSITION 3.22

Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $f : X \to [0,\infty]$ be a measurable function. Then $\int_X f\ d\mu = 0$ if and only if $f = 0$ a.e.

PROOF. If $f = 0$ a.e., then by Proposition 3.21, f is integrable and

$$\int_X f\ d\mu = \int_X 0\ d\mu = 0\cdot\mu(X) = 0.$$

Conversely, suppose $\int_X f\ d\mu = 0$. Then by Markov's inequality (Exercise 3.2),

$$\mu(\{f > c\}) \le \frac{1}{c}\int_X f\ d\mu = 0$$

for every $c > 0$. Therefore, by continuity of μ from below,

$$\mu(\{f \ne 0\}) = \mu\left(\bigcup_{n\in\mathbb{N}}\left\{f > \frac{1}{n}\right\}\right) = \lim_{n\to\infty}\mu\left(\left\{f > \frac{1}{n}\right\}\right) = 0.$$

That is, $f = 0$ a.e. □

The examples above (especially Proposition 3.21) show that null sets are negligible from the point of view of integration, and we can very often ignore modifications that happen on null sets. There is one subtle issue that requires care, however: in general, a subset of a null set may not be measurable and non-measurable modifications on null sets may create issues. For this reason, it is often convenient to work with *complete* measure spaces, as defined below.

Definition 3.23

A measure space $(X, \mathcal{B}, \mu)$ is *complete* if every subset of every null set is measurable. That is, if $E \subseteq X$ and there exists $N \in \mathcal{B}$ with $E \subseteq N$ and $\mu(N) = 0$, then $E \in \mathcal{B}$.

The following proposition is a useful tool for passing to complete measure spaces.

Proposition 3.24

Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $\mathcal{N} = \{N \in \mathcal{B} : \mu(N) = 0\}$ be the σ-ideal of μ-null sets. Then the family $\overline{\mathcal{B}} = \{E \cup F : E \in \mathcal{B}, F \subseteq N \in \mathcal{N}\}$ is a σ-algebra, and there is a unique extension $\overline{\mu}$ of μ to $\overline{\mathcal{B}}$.

Proof. Exercise 3.9. □

Definition 3.25

The *completion* of a measure space $(X, \mathcal{B}, \mu)$ is the space $(X, \overline{\mathcal{B}}, \overline{\mu})$, where $\overline{\mathcal{B}}$ and $\overline{\mu}$ are as defined in Proposition 3.24.

6. The Dominated Convergence Theorem

We have already seen two fundamental convergence theorems for integration against a measure: the monotone convergence theorem and Fatou's lemma. We are nearly ready to state another fundamental result about integration: the dominated convergence theorem. First, we need to introduce the two notions of convergence that will be related by the dominated convergence theorem.

Definition 3.26

Let $(X, \mathcal{B}, \mu)$ be a measure space.

- We say that a sequence $(f_n)_{n\in\mathbb{N}}$ of functions on X *converges almost everywhere* to a function f if $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in X$.
- A sequence $(f_n)_{n\in\mathbb{N}}$ of integrable functions *converges in L^1* to $f \in L^1(\mu)$ if
$$\|f_n - f\|_1 = \int_X |f_n - f| \, d\mu \to 0$$
in $\mathbb{R}$ as $n \to \infty$.

The dominated convergence theorem says that any sequence that converges almost everywhere and is "L^1-dominated" will converge in L^1. The precise mathematical formulation is as follows:

Theorem 3.27: Dominated Convergence Theorem

Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions, $f_n : X \to \mathbb{C}$, and let $f : X \to \mathbb{C}$ be measurable. Suppose

(1) $f_n \to f$ a.e., and
(2) there is an integrable function $g : X \to [0, \infty)$ such that $\sup_{n\in\mathbb{N}} |f_n| \le g$ a.e.

Then f is integrable and $f_n \to f$ in $L^1(\mu)$. In particular,

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$$

Proof. First, $|f| \leq |g|$ a.e., so f is integrable.

Observe:

$$\begin{aligned}\int_X 2g \, d\mu - \limsup_{n\to\infty} \int_X |f - f_n| \, d\mu &= \liminf_{n\to\infty} \int_X (2g - |f - f_n|) \, d\mu \\ &\geq \int_X \liminf_{n\to\infty} (2g - |f - f_n|) \, d\mu && \text{(Fatou's lemma)} \\ &= \int_X 2g \, d\mu && (f_n \to f)\end{aligned}$$

Rearranging, we conclude

$$\limsup_{n\to\infty} \int_X |f - f_n| \, d\mu \leq 0.$$

Using the triangle inequality for the integral,

$$\left| \int_X f \, d\mu - \int_X f_n \, d\mu \right| \leq \int_X |f - f_n| \, d\mu \to 0,$$

so

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$$

□

The assumption that the sequence $(f_n)_{n\in\mathbb{N}}$ is "dominated" by an integrable function g is a necessary assumption to avoid "escape of mass to infinity," as the following example demonstrates.

Example 3.28

Let $X = \mathbb{Z}$, $\mathcal{B} = \mathscr{P}(\mathbb{Z})$, and let μ be the counting measure. Let $f_n = \mathbb{1}_{\{n\}}$. Then $f_n(x) \to 0$ for every $x \in X$. However,

$$\int_X f_n \, d\mu = 1$$

for every $n \in \mathbb{N}$, while

$$\int_X \lim_{n\to\infty} f_n \, d\mu = \int_X 0 \, d\mu = 0 \neq 1.$$

Additional Reading

For other presentations of integration on abstract measures spaces, see [**1**, Section 2.1–2.3], [**4**, Chapter 1], [**5**, Sections 2.1 and 6.2], and/or [**6**, Section 1.3 and Subsection 1.4.4]. The development of integration in the books of Folland [**1**] and Rudin [**4**] is very similar to the presentation in these notes. By contrast, Stein and Shakarchi [**5**] and Tao [**6**] first develop integration in the special case of the Lebesgue measure before moving to abstract spaces. The book of Stein and Shakarchi [**5**] also proves the fundamental convergence theorems in a different order, starting with a special case of the dominated convergence theorem known as the *bounded convergence theorem*, and then deducing Fatou's lemma, the monotone convergence theorem, and the general case of the dominated convergence theorem.

There is a very nice book of Oxtoby [3] that develops useful analogies between measure spaces and topological spaces and includes a discussion of null sets in relation to a σ-ideal of "topologically negligible" sets called *meager* sets or sets of *first category*.

Exercises

3.1 Prove the Borel–Cantelli lemma: if $(A_n)_{n\in\mathbb{N}}$ is a family of measurable subsets of a probability space $(X, \mathcal{B}, \mu)$ and $\sum_{n\in\mathbb{N}} \mu(A_n) < \infty$, then

$$\mu\left(\{x \in X : x \in A_n \text{ for infinitely many } n \in \mathbb{N}\}\right) = 0.$$

3.2 Let $(X, \mathcal{B}, \mu)$ be a measure space and f a measurable function. Prove Markov's inequality: for any $c > 0$,

$$\mu\left(\{|f| \geq c\}\right) \leq \frac{1}{c}\int_{\{|f|>c\}} |f| \ d\mu \leq \frac{1}{c}\int_X |f| \ d\mu.$$

3.3 Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ be measure space, and let $T : X \to Y$ be a measurable function. Define $T\mu : \mathcal{C} \to [0, \infty]$ by $(T\mu)(A) = \mu(T^{-1}(A))$. Prove $T\mu = \nu$ if and only if for every integrable function $f : Y \to \mathbb{C}$,

$$\int_Y f \ d\nu = \int_X f \circ T \ d\mu.$$

3.4 Let $(X, \mathcal{B}, \mu)$ be a probability space. Let $(A_n)_{n\in\mathbb{N}}$ be a family of measurable sets with $a = \inf_{n\in\mathbb{N}} \mu(A_n) > 0$. Show that there is a set $E \subseteq \mathbb{N}$ such that $\overline{d}(E) := \limsup_{N\to\infty} \frac{|E\cap\{1,\dots,N\}|}{N} \geq a$, and for any finite set $F \subseteq E$, $F \neq \emptyset$, one has $\mu\left(\bigcap_{n\in F} A_n\right) > 0$ by proving the following intermediate steps:

(a) Justify that we can assume without loss of generality that $\bigcap_{n\in F} A_n \neq \emptyset$ if and only if $\mu\left(\bigcap_{n\in F} A_n\right) > 0$ for every finite set $F \subseteq \mathbb{N}$. It may help to define the countable set

$$\mathcal{F} = \left\{F \subseteq \mathbb{N} : |F| < \infty, \bigcap_{n\in F} A_n \neq \emptyset, \text{and } \mu\left(\bigcap_{n\in F} A_n\right) = 0\right\}.$$

(b) Prove

$$\int_X \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_{A_n} \ d\mu \geq a.$$

(c) Define $E = \{n \in \mathbb{N} : x \in A_n\}$ for a suitable choice of $x \in X$.

3.5 Let $(X, \mathcal{B}, \mu)$ be a measure space. Suppose $f : X \to [0, \infty]$ is a measurable function. Define $\nu : \mathcal{B} \to [0, \infty]$ by

$$\nu(E) = \int_E f \ d\mu.$$

Prove that ν is a measure.

3.6 Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $f : X \to \mathbb{C}$ be an integrable function. Prove that for any $\varepsilon > 0$, there exists $\delta > 0$ with the following property: if $E \in \mathcal{B}$ and $\mu(E) < \delta$, then $\left|\int_E f \ d\mu\right| < \varepsilon$.

3.7 Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $f, g : X \to \mathbb{C}$ be integrable functions. Show that $f = 0$ a.e. if and only if $\int_E f \ d\mu$ for every $E \in \mathcal{B}$.

3.8 Show that a measure space $(X, \mathcal{B}, \mu)$ is complete if and only if it satisfies the following property: for functions $f, g : X \to \mathbb{C}$, if f is measurable and $f = g$ a.e., then g is measurable.

3.9 Prove Proposition 3.24.

3.10 Show that simple functions are dense in L^1. That is, if $(X, \mathcal{B}, \mu)$ is a measure space and $f \in L^1(\mu)$, then for every $\varepsilon > 0$, there exists a simple function $s : X \to \mathbb{C}$ such that $\|f - s\|_1 < \varepsilon$.

3.11 Let $(X, \mathcal{B}, \mu)$ be a measure space and $E \in \mathcal{B}$. If $(E_n)_{n \in \mathbb{N}}$ is a sequence of measurable sets and $E = \bigcup_{n \in \mathbb{N}} E_n$, prove that for every integrable function $f \in L^1(\mu)$,

$$\lim_{n \to \infty} \int_{E_n} f \; d\mu = \int_E f \; d\mu.$$

State and prove an analogous result for decreasing sequences.

3.12 Prove

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = \sum_{k \geq 0} \frac{x^k}{k!}.$$

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